Collection of Formulas

Electromagnetic Fields EITF80

Department of Electrical and Information Technology

Lund University, Sweden

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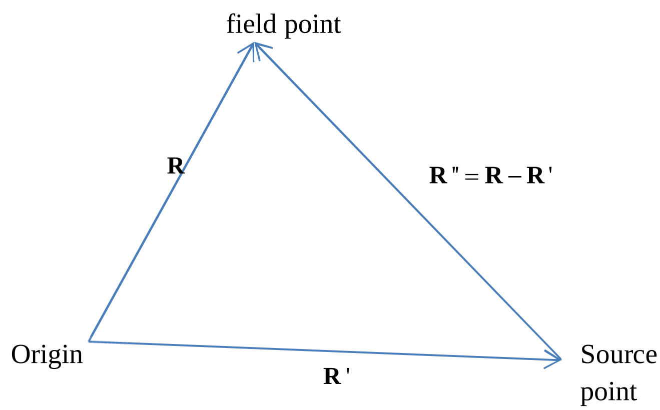
ELECTROSTATICS



Coulomb's Law

The force F on a point charge q_1 at point **R**, resulting from a point charge q at point $\mathbf{R}'$

$$\mathbf{F} = \frac{q_1 q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R, \text{ where } R = |\mathbf{R}''| = |\mathbf{R} - \mathbf{R}'| \text{ and } \mathbf{a}_R = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

Electric Field Intensity E in Vacuum

From point charge q at $\mathbf{R}'$ $\qquad \mathbf{E}(\mathbf{R}) = \frac{q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$

From charge distribution $\qquad \mathbf{E}(\mathbf{R}) = \int \frac{1}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \, dq(\mathbf{R}')$

$$\text{where } dq(\mathbf{R}') = \begin{cases} \rho \, dv & \text{for volume charge density } \rho \\ \rho_s \, ds & \text{for surface charge density } \rho_s \\ \rho_l \, dl & \text{for line charge density } \rho_l \end{cases}$$

From point dipole $\mathbf{p} = p\mathbf{a}_z$ $\qquad \mathbf{E}(\mathbf{R}) = \frac{p}{4\pi\varepsilon_0 R^3}(2\cos\theta \, \mathbf{a}_R + \sin\theta \, \mathbf{a}_\theta)$

From line charge (of density ρ_l) $\qquad \mathbf{E}(\mathbf{R}) = \frac{\rho_l}{2\pi\varepsilon_0 r}\mathbf{a}_r$

Force F on point charge q

$$\mathbf{F} = \begin{cases} q\mathbf{E} & \text{for electrostatics} \\ q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) & \text{for general case} \end{cases}$$

Electric Potential V

From point charge q at $\mathbf{R}'$ $$V(\mathbf{R}) = \frac{q}{4\pi\varepsilon_0 R}$$

From charge distribution $$V(\mathbf{R}) = \int \frac{1}{4\pi\varepsilon_0 R} dq(\mathbf{R}')$$

From point dipole $\mathbf{p} = p\mathbf{a}_z$ $$V(\mathbf{R}) = \frac{\mathbf{p} \bullet \mathbf{a}_R}{4\pi\varepsilon_0 R^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 R^2}$$

From line charge (of density ρ_l) $$V(\mathbf{R}) = \frac{\rho_l}{2\pi\varepsilon_0} \ln\frac{1}{r}$$

$$\mathbf{E} = -\nabla V \quad \text{for electrostatics}$$

Electric Flux Density D

Gauss' Law $$\oint \mathbf{D} \bullet \mathbf{a}_n ds = \int \rho \, dv$$

where $\mathbf{a}_n$ is the outward unit vector normal to the surface of the volume.

Polarization P $$\mathbf{P} = \frac{d\mathbf{p}}{dv}$$

Surface polarization charge density $$\rho_{ps} = \mathbf{P} \bullet \mathbf{a}_n$$

Volume polarization charge density $$\rho_p = -\nabla \bullet \mathbf{P}$$

Relationship between **P**, **E** and **D** $$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} & \text{for general case} \\ \mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} \end{cases}$$

Boundary Conditions $$\begin{cases} E_t \text{ is continuous} \\ \rho_s = \mathbf{a}_{n2} \bullet (\mathbf{D}_1 - \mathbf{D}_2) \end{cases}$$

where ρ_s is the surface charge density of free charges, and a_{n2} is the normal unit vector pointing from region 2 toward region 1.

STEADY ELECTRIC CURRENTS

Current density J $I = \int \mathbf{J} \bullet \mathbf{a}_n \, ds$

Equation of Continuity (from principle of conservation of charge)

In differential form $\nabla \bullet \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

In integral form $\oint \mathbf{J} \bullet \mathbf{a}_n \, ds = -\frac{dQ}{dt}$

Conductivity σ $\mathbf{J} = \sigma \mathbf{E}$

Total Power Dissipation P $\mathbf{P} = \int \mathbf{J} \bullet \mathbf{E} \, dv$

Boundary Conditions

$$\begin{cases} \mathbf{J}_s = \mathbf{a}_{n2} \bullet (\mathbf{J}_1 - \mathbf{J}_2) & (\text{for no surface current } \mathbf{J}_s = 0) \\ \mathbf{E}_t \text{ is continuous} \end{cases}$$

Time Constant $\tau = RC = \frac{\varepsilon_r \varepsilon_0}{\sigma}$

Analogy between Electrostatics and Steady Electric Currents

$$\begin{aligned} \mathbf{E}, V &\leftrightarrow \mathbf{E}, V \\ \mathbf{D} &\leftrightarrow \mathbf{J} \\ \varepsilon_r \varepsilon_0 &\leftrightarrow \sigma \\ Q &\leftrightarrow I \\ C &\leftrightarrow G \end{aligned}$$

MAGNETOSTATICS

Magnetic Flux Density B in Vacuum

From point dipole $\mathbf{m} = m\mathbf{a}_z$ $\quad$ $\mathbf{B}(\mathbf{R}) = \dfrac{\mu_0 m}{4\pi R^3}\left(2\cos\theta\mathbf{a}_R + \sin\theta\mathbf{a}_\theta\right)$

From current density $\mathbf{J}(\mathbf{R}')$ $\quad$ $\mathbf{B}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}(\mathbf{R}')\times\mathbf{a}_R}{R^2}\,dv'$

From current path $\quad$ $\mathbf{B}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I\,d\boldsymbol{l}'\times\mathbf{a}_R}{R^2}$

From circular wire loop $\quad$ $\mathbf{B}(x=0, y=0, z) = \dfrac{\mu_0 I\, b^2}{2\left(b^2+z^2\right)^{3/2}}\mathbf{a}_z$

From long straight wire path $\quad$ $\mathbf{B}(\mathbf{R}) = \dfrac{\mu_0 I}{2\pi r}\mathbf{a}_\phi$

Vector Magnetic Potential A in Vacuum

From point dipole $\mathbf{m}$ $\quad$ $\mathbf{A}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{a}_R}{R^2}$

From current density $\mathbf{J}(\mathbf{R}')$ $\quad$ $\mathbf{A}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}(\mathbf{R}')}{R}\,dv'$

From current path $\quad$ $\mathbf{A}(\mathbf{R}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I\,d\boldsymbol{l}'}{R}$

Magnetic Flux Φ $\quad$ $\Phi = \displaystyle\int \mathbf{B}\bullet\mathbf{a}_n\,ds = \oint \mathbf{A}\bullet d\boldsymbol{l}$

Flux Linkage $\boldsymbol{\Lambda}$ $\quad$ $\boldsymbol{\Lambda} = N\Phi$

Self-inductance L **and Mutual-inductance** M

$$\boldsymbol{\Lambda}_1 = L_1 I_1 + M I_2$$
$$\boldsymbol{\Lambda}_2 = L_2 I_2 + M I_1$$

Magnetic Field Intensity H

Ampere's Law

$$\oint \mathbf{H} \bullet d\boldsymbol{l} = \int \mathbf{J} \bullet \mathbf{a}_n ds = I_{\text{enclosed}}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Relationship between magnetization vector **M**, **B** and **H**

$$\begin{cases} \mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right) & \text{for general case} \\ \mathbf{B} = \mu_r \mu_0 \mathbf{H} \end{cases}$$

Boundary Conditions

$$\begin{cases} \mathbf{a}_{n2} \times \left(\mathbf{H}_1 - \mathbf{H}_2 \right) = \mathbf{J}_s \\ B_n \text{ is continuous} \end{cases}$$

where $\mathbf{J}_s$ is the current density of free surface current, and a_{n2} is the normal unit vector pointing from region 2 toward region 1.

Reluctance

$$R = \frac{l}{\mu_r \mu_0 S}$$

Magnetic Force

$$d\mathbf{F}_m = I\, d\boldsymbol{l} \times \mathbf{B}$$

Magnetic Moment m for Current Loop

$$\mathbf{m} = \int I\, \mathbf{a}_n\, ds$$

Torque $\mathbf{T}_m$ on Magnetic Dipole m

$$\mathbf{T}_m = \mathbf{m} \times \mathbf{B}$$

Force on Magnetic Dipole m

$$\mathbf{F} = \left(\mathbf{m} \bullet \nabla \right) \mathbf{B} + \mathbf{m} \times \left(\nabla \times \mathbf{B} \right)$$

ELECTROMAGNETIC FIELD

Faraday's Law of Electromagnetic Induction

$$v = RI = -\frac{d\Phi}{dt}$$

Induced emf v

$$v = \oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \bullet d\boldsymbol{l}$$

$$v = -\frac{d\Lambda}{dt} \text{ for coils}$$

Faraday's Law of Electromagnetic Induction

In differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In integral form

$$\oint \mathbf{E} \bullet d\boldsymbol{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{a}_n \, ds$$

Maxwell's Equations

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \bullet \mathbf{D} = \rho \\ \nabla \bullet \mathbf{B} = 0 \end{cases}$$

Electromagnetic Constants

$$\mu_0 = 4\pi \cdot 10^{-7} \,\text{H/m}, \quad \varepsilon_0 \approx \frac{10^{-9}}{36\pi} \,\text{F/m}, \quad c \approx 3 \cdot 10^8 \,\text{m/s}$$

$$\frac{1}{\mu_0 \varepsilon_0} = c^2, \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \quad \eta_0 \approx 120\pi \,\Omega \approx 377 \,\Omega$$

Potentials

Relationship between vector magnetic potential **A** and magnetic flux density **B**

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Relationship between scalar electric potential V, vector magnetic potential **A** and electric field intensity **E**

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Poynting Vector

$$P(\mathbf{R}, t) = \mathbf{E}(\mathbf{R}, t) \times \mathbf{H}(\mathbf{R}, t)$$

$$P_{\text{av}}(\mathbf{R}) = \frac{1}{2} Re\left[\mathbf{E}(\mathbf{R}) \times \mathbf{H}^*(\mathbf{R})\right] \text{ (time-harmonic signal)}$$

TIME-HARMONIC FIELD

Plane Wave

$\mathbf{E} = \hat{E}\cos(\omega t - \mathbf{k}\bullet\mathbf{R} + \phi)\mathbf{a}_E$ instantaneous value

$\mathbf{E} = E_0 e^{-jk\bullet R}\mathbf{a}_E$ complex form

$E_0 = \hat{E}\, e^{j\phi}$ peak value

$E_0 = \dfrac{\hat{E}}{\sqrt{2}} e^{j\phi}$ root-mean-square (rms) value

Group Velocity

$$\upsilon = \frac{1}{\sqrt{\mu_r \mu_0 \varepsilon_r \varepsilon_0}}, \quad \upsilon = \frac{\omega}{k}, \quad k = |\mathbf{k}|$$

Characteristic Impedance

$$\eta = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}}$$

Right Hand Rule

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H, \quad E = \eta H, \quad \mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_B, \quad E = \upsilon B$$

VECTOR IDENTITIES

$$\mathbf{A}\bullet(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\bullet(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\bullet(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\bullet\mathbf{C}) - \mathbf{C}(\mathbf{A}\bullet\mathbf{B})$$

Divergence Theorem

$$\int_{\text{Volume}} \nabla\bullet\mathbf{A}\, dv = \oint_{\text{Surface}} \mathbf{A}\bullet\mathbf{a}_n\, ds$$

Stoke’s Theorem

$$\int_{\text{Surface}} (\nabla\times\mathbf{A})\bullet\mathbf{a}_n\, ds = \oint_{\text{Curve}} \mathbf{A}\bullet d\boldsymbol{l}$$

Null Identities

$$\nabla\times(\nabla V) = 0$$

$$\nabla\bullet(\nabla\times\mathbf{A}) = 0$$

COORDINATE SYSTEM

Cartesian Coordinates (x, y, z)

Position vector $\mathbf{R} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$

Line element $d\boldsymbol{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$

Volume element $dv = dx\,dy\,dz$

Surface elements $ds_x = dy\ dz$

$ds_y = dx\ dz$

$ds_z = dx\ dy$

Differential operators $\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$

$$\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

Position vector $\mathbf{R} = r\,\mathbf{a}_r + z\,\mathbf{a}_z$

Unit vectors $\mathbf{a}_r = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$

$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

Line element $d\boldsymbol{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z$

Volume element $dv = r\,dr\,d\phi\,dz$

Surface elements $ds_r = r\ d\phi\ dz$

$ds_\phi = dr\ dz$

$ds_z = r\ dr\ d\phi$

Differential operators $\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$

$$\nabla \bullet \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,A_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_r \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{a}_z \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,A_\phi\right) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates (R,θ,ϕ)

Position vector $\mathbf{R} = R\,\mathbf{a}_R$

Unit vectors

$$\mathbf{a}_R = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$$

Line element $d\boldsymbol{l} = dR\,\mathbf{a}_R + R\,d\theta\,\mathbf{a}_\theta + R\sin\theta\,d\phi\,\mathbf{a}_\phi$

Volume element $dv = R^2 \sin\theta\,dR\,d\theta\,d\phi$

Surface elements

$$ds_R = R^2 \sin\theta\, d\theta\, d\phi$$

$$ds_\theta = R \sin\theta\, dR\, d\phi$$

$$ds_\phi = R\, dR\, d\theta$$

Differential operators

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \bullet \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 A_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_R \frac{1}{R\sin\theta}\left(\frac{\partial}{\partial \theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial \phi}\right) + \mathbf{a}_\theta \frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}\left(R\,A_\phi\right)\right]$$
$$+ \mathbf{a}_\phi \frac{1}{R}\left(\frac{\partial}{\partial R}\left(R\,A_\theta\right) - \frac{\partial A_R}{\partial \theta}\right)$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad , n \neq -1$$

$$\int \frac{1}{x} dx = \ln x$$

$$\int \sqrt{x^2 + a^2} dx = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2} \right)\right]$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

$$\int \frac{1}{\left(x^2 + a^2 \right)^{3/2}} dx = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a}$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a}$$

$$\int \frac{1}{\cos^2 x} dx = \tan x$$

$$\int \frac{1}{\sin x} dx = \ln \left| \tan \frac{x}{2} \right|$$

$$\int \ln x \, dx = x \ln x - x$$

TRIGONOMETRIC IDENTITIES

$$\cos\left(\alpha - \beta\right) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$\cos\left(\alpha + \beta\right) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\sin\left(\alpha - \beta\right) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

$$\sin\left(\alpha + \beta\right) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\begin{cases} a\sin t + b\cos t = \sqrt{a^2 + b^2}\,\sin\left(t + \varphi\right) \\ \sin\varphi = \dfrac{b}{\sqrt{a^2 + b^2}}, \cos\varphi = \dfrac{a}{\sqrt{a^2 + b^2}} \end{cases}$$