Trade-off between Q-factor and efficiency for small antennas

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Based on 'Trade-off Between Antenna Efficiency and Q-Factor' (TEAT-7260)

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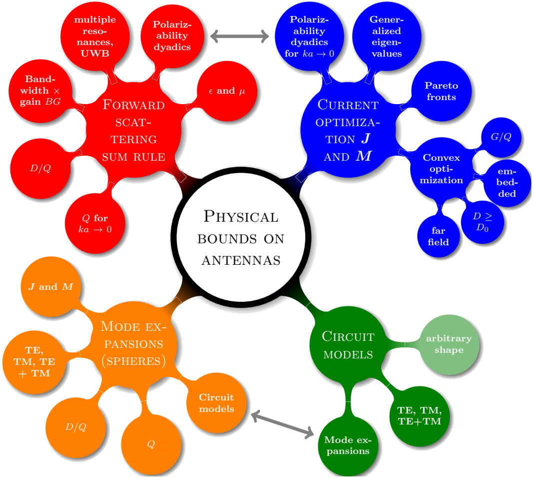
Antenna challenges and limitations

Often good to have antennas with all or some of

- low Q-factor (high bandwidth)
- low losses
- high directivity
- high capacity (diversity)
- matching

Many approaches to determine physical bounds. Most general approaches by optimization over antenna current density.

What can be said about the trade-off between different parameters?



Trade-off between bandwidth and efficiency

Trade-off between bandwidth and efficiency can be analyzed by the optimization problems

$$\begin{aligned} &\text{maximize} && \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} \leq 2\bar{P}_{\mathrm{w}} \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} \leq 2\bar{P}_{\mathrm{w}} \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\Omega}\mathbf{I} \leq 2\bar{P}_{\Omega}. \end{aligned} \qquad \text{(T)}$$

$$\begin{aligned} &\text{maximize} && \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \\ &\text{subject to} && \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I} = 2\bar{P}_{\mathrm{w}} \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I} = 2\bar{P}_{\mathrm{w}} \\ &&& \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\Omega}\mathbf{I} \leq 2\bar{P}_{\Omega}. \end{aligned} \qquad \text{(S)}$$

Self-resonance is enforced by equal electric and magnetic stored energies. Optimizing over the current $\mathbf{I}$ ($N \times 1$-matrix) with given positive semidefinite $N \times N$ matrices $\mathbf{R}_{\mathrm{r}}$ (radiated power), $\mathbf{X}_{\mathrm{m}}$ (stored magnetic energy), $\mathbf{X}_{\mathrm{e}}$ (stored electric energy),$\mathbf{R}_{\Omega}$ (ohmic losses).

These QCQPs are not convex so we need to reformulate them in convex (or some other solvable) form.

Pareto in α and relaxation in ν

Start with by forming linear combinations between the constraints as Pareto in α and relaxation in ν

$$\begin{aligned} \text{maximize}_{\mathbf{I}} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I} \\ \text{subject to} \quad & \mathbf{I}^{\mathsf{H}}\mathbf{X}_{\alpha\nu}\mathbf{I} \leq 2(\alpha\bar{P}_{\mathrm{w}} + (1-\alpha)\bar{P}_{\Omega}), \end{aligned} \tag{R}$$

where $\alpha \in [0,1]$, $\nu \in [0,1]$ (or $\nu \in \mathbb{R}$) and

$$\mathbf{X}_{\alpha\nu} = \alpha\nu\mathbf{X}_{\mathrm{e}} + \alpha(1-\nu)\mathbf{X}_{\mathrm{m}} + (1-\alpha)\mathbf{R}_{\Omega}$$

This is a Rayleigh quotient with the solution (for $\mathbf{X}_{\alpha\nu} \succeq \mathbf{0}$ and fixed α)

$$\text{maximize}_{\nu} \ \min \operatorname{eig}(\mathbf{X}_{\alpha\nu}, \mathbf{R}_{\mathrm{r}}), \tag{E}$$

where eig denotes the set of eigenvalues $\{\gamma_n\}$ which solves $\mathbf{X}_{\alpha\nu}\mathbf{I}_n = \gamma_n\mathbf{R}_{\mathrm{r}}\mathbf{I}_n$.

The solution $\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}$ to (R) (and (15)) is greater than or equal to the corresponding solutions of (T) and (S).

Line search, duality gap, and derivative

The relaxed problem (R) is concave in ν (for fixed α) and solved by a line search over ν. Derivative of the smallest eigenvalue (assuming non-degenerate eigenvalues)

$$\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}\nu} = \frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}'_{\alpha\nu}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = -\alpha\frac{\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} \begin{cases} < 0 & \text{if inductive } \mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} > 0 \\ = 0 & \text{if resonant } \mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} = 0 \\ > 0 & \text{if capacitive } \mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} < 0 \end{cases}$$

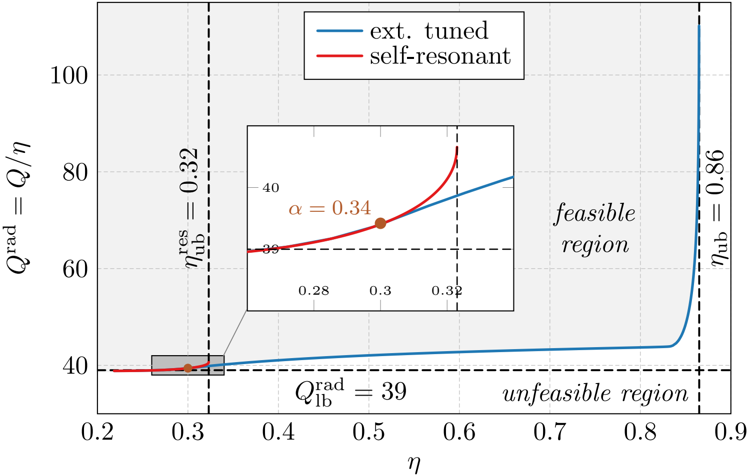
Degenerate eigenvalues are connected to geometrical symmetries and we can decompose the problem into orthogonal subspaces with non-degenerate eigenvalues. A few different cases, consider *e.g.*, the case with the optimal ν in the inner region. Then $\gamma' = 0$ hence $\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} = 0$ implying that the constraints in (S) are satisfied and hence there is no duality gap.

Pareto front η vs $Q^{\text{rad}} = Q/\eta$

externally tuned
High cost in Q for η_{ub}.
self-resonant
Low cost in Q for $\eta_{\text{ub}}^{\text{res}}$

Often simpler to use the dissipation factor

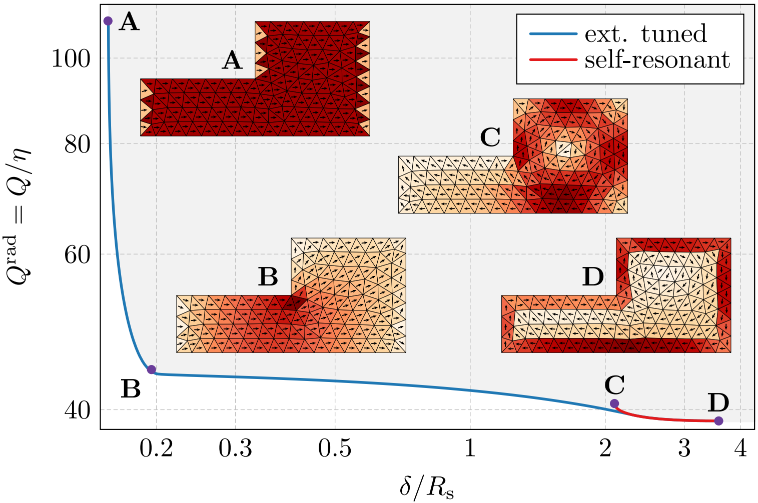
$$\delta = \frac{P_{\text{loss}}}{P_{\text{rad}}} = \eta^{-1} - 1$$



Self-resonance has a high cost for efficiency.

Pareto front $\delta = P_{\text{loss}}/P_{\text{rad}}$ vs $Q^{\text{rad}} = Q/\eta$

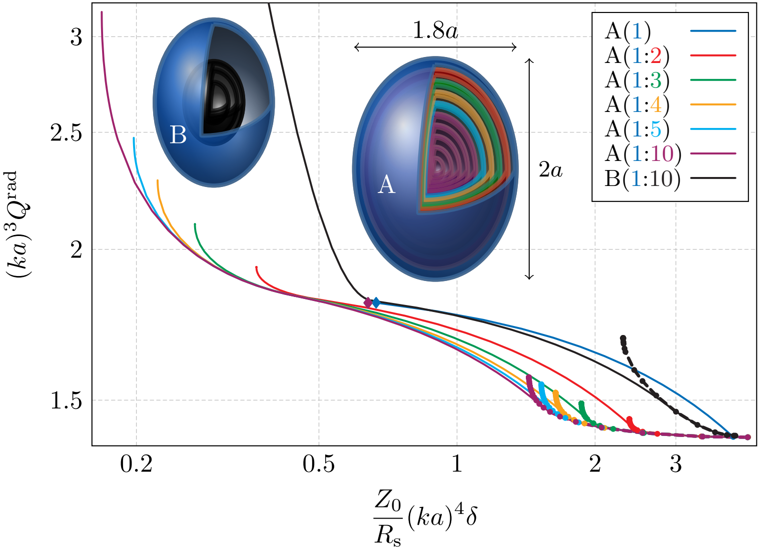
- D $\min Q^{\text{rad}}$ from loop current at the rim (M-dipole) and charges at two edges (E-dipole).
- C $\min \delta$ (self-res.) from distributed loop current.
- B Electric dipole (not self-res.).
- A $\min \delta$ from homogeneous current density.



Pareto curves for layered prolate spheroids

Tuned (solid) and self resonant (dotted curves).

- Sufficient with surface currents for the radiation Q-factor [GCJ12].
- Volumetric current densities can reduce the dissipation factor δ (increased efficiency).

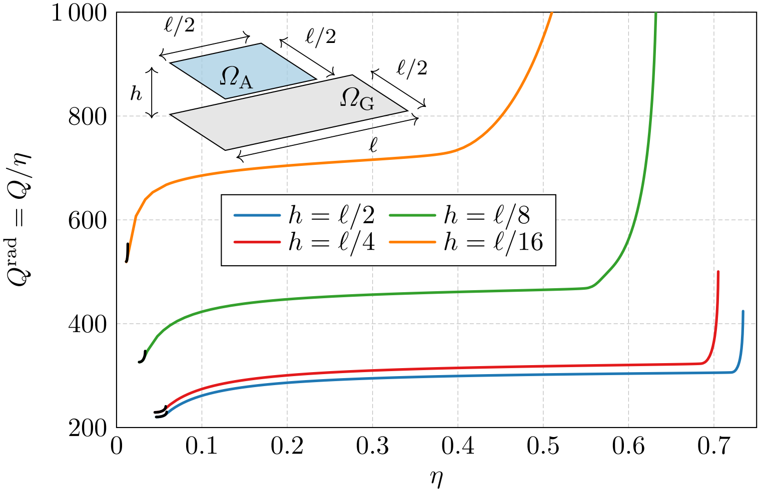


Pareto fronts for controllable regions

Similar results for cases with controllable currents in subregions.

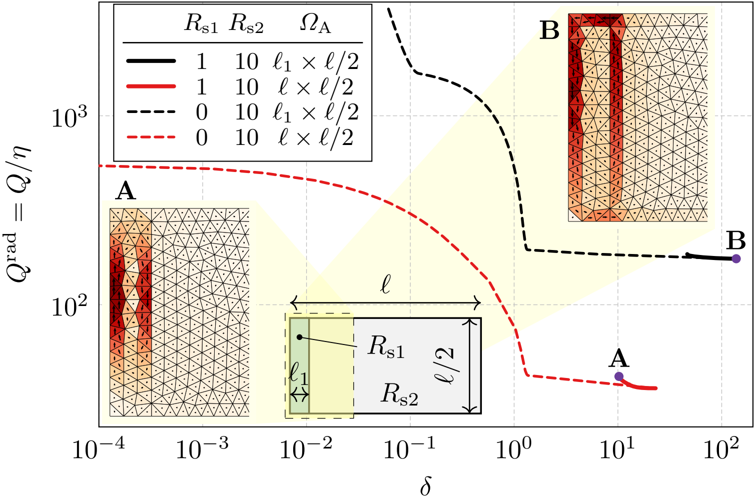
Ω_{A}: antenna region with controllable currents.

Ω_{G}: ground plane with induced currents.



Pareto fronts for regions with inhomogeneous resistivity

Large differences between $\min Q^{\mathrm{rad}}$ and $\min\delta$ for structures with inhomogeneous resistivity.
$\min\delta$ by reducing currents in the lossy region.



Limiting cases

The limiting cases for the Pareto front $\alpha = 0, 1$ can be solved directly. The minimum dissipation factor results in the eigenvalue problem [Har60; JC17]

$$(\eta_{\mathrm{ub}})^{-1} - 1 = \delta_{\mathrm{lb}} = \min_{\mathbf{I}} \frac{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\Omega}\mathbf{I}}{\mathbf{I}^{\mathsf{H}}\mathbf{R}_{\mathrm{r}}\mathbf{I}} = \min \operatorname{eig}(\mathbf{R}_{\Omega}, \mathbf{R}_{\mathrm{r}}).$$

Self resonance is enforced by adding $(\mathbf{I}^{\mathsf{H}}\mathbf{X}\mathbf{I} = 0)$ to the optimization problem and solved as (see also [JC17; Pfe17; Tha18])

$$(\eta_{\mathrm{ub}}^{\mathrm{res}})^{-1} - 1 = \delta_{\mathrm{lb}}^{\mathrm{res}} = \max_{\nu\in\mathbb{R}} \min \operatorname{eig}(\nu\mathbf{X} + \mathbf{R}_{\Omega}, \mathbf{R}_{\mathrm{r}}).$$

Note that these optimization/eigenvalue problems are easy to solve. They can also be further simplified by the factorization $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$ discussed later.

Limit of electrically small antennas

TM and TE eigenvalue problems

$$k^3 Q^{\mathrm{rad}}_{\mathrm{lb,TM}} = \min_{\nabla\times} \operatorname{eig}(\mathbf{C}_{\mathrm{i}}, \mathbf{R}_{\mathrm{TM}}) = \frac{6\pi}{\max \operatorname{eig} \boldsymbol{\gamma}_{\mathrm{e}}}, \quad k^3 Q^{\mathrm{rad}}_{\mathrm{lb,TE}} = \min_{\nabla\cdot} \operatorname{eig}(\mathbf{L}, \mathbf{R}_{\mathrm{TE}}) = \frac{6\pi}{\max \operatorname{eig} \boldsymbol{\gamma}_{\mathrm{m}}}$$

are used to determine the mixed TM-TE Q-factor

$$Q^{\mathrm{rad}}_{\mathrm{lb}} = \frac{6\pi}{k^3(\max \operatorname{eig} \boldsymbol{\gamma}_{\mathrm{e}} + \max \operatorname{eig} \boldsymbol{\gamma}_{\mathrm{m}})}.$$

The inductor Q-factor

$$\frac{Q_{\mathrm{ub,L}}}{k} = \max \frac{\mathbf{I}_0^{\mathsf{H}} \mathbf{L} \mathbf{I}_0}{\mathbf{I}_0^{\mathsf{H}} \mathbf{R}_\Omega \mathbf{I}_0} = \max_{\nabla\cdot} \operatorname{eig}(\mathbf{L}, \mathbf{R}_\Omega).$$

is used to express the minimal TM dissipation factor

$$\delta_{\mathrm{lb,TM}} = \frac{\min \operatorname{eig}_{\nabla\times}(\mathbf{C}_{\mathrm{i}}, \mathbf{R}_{\mathrm{TM}})}{k^4 \max \operatorname{eig}_{\nabla\cdot}(\mathbf{L}, \mathbf{R}_\Omega)} = \frac{6\pi}{k^3 \gamma Q_{\mathrm{ub,L}}} = \frac{Q^{\mathrm{rad}}_{\mathrm{lb,TM}}}{Q_{\mathrm{ub,L}}}.$$

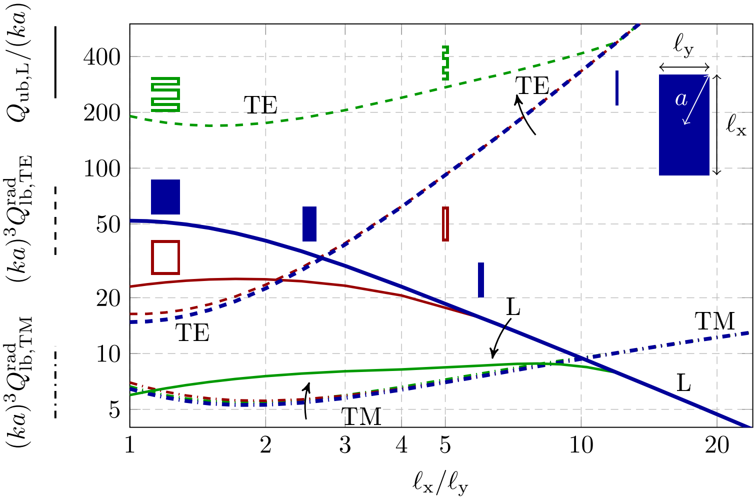
Normalized Q-factors for planar rectangle with sides ℓ_{x} and ℓ_{y}

Lower bound on the dissipation factor

$$\delta_{\mathrm{lb,TM}} = \frac{Q^{\mathrm{rad}}_{\mathrm{lb,TM}}}{Q_{\mathrm{ub,L}}}.$$

Want Low $Q^{\mathrm{rad}}_{\mathrm{lb,TM}}$ and high $Q_{\mathrm{ub,L}}$.

Arrows show results of removing material (structure). Here, from solid rectangles to loops and Meanderlines.



Pareto front $\delta = P_{\text{loss}}/P_{\text{rad}}$ vs $Q^{\text{rad}} = Q/\eta$

- D $\min Q^{\text{rad}}$ from loop current at the rim (M-dipole) and charges at two edges (E-dipole).
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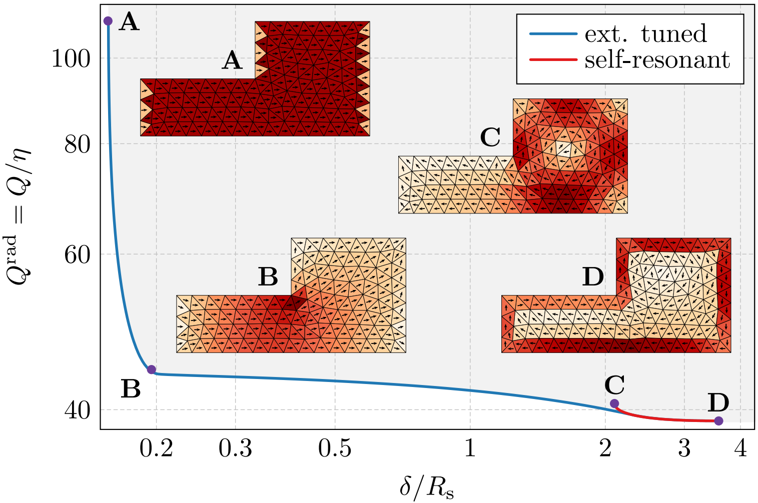


Computational cost

The Pareto fronts are determined by solving

$$\mathrm{maximize}_{\nu} \ \ \min \mathrm{eig}(\mathbf{X}_{\alpha\nu}, \mathbf{R}_{\mathrm{r}}),$$

which are concave in ν and $0 < \alpha < 1$ determines the Pareto front from

$$\mathbf{X}_{\alpha\nu} = \alpha\nu\mathbf{X}_{\mathrm{e}} + \alpha(1-\nu)\mathbf{X}_{\mathrm{m}} + (1-\alpha)\mathbf{R}_{\Omega}$$

- Need to sample α in 100-200 points for a smooth curve.
- 10-30 evaluations for $\max_{\nu}$ (bisection algorithm).

In total 10^4 to 10^5 evaluations of generalized eigenvalue problems (similar to characteristic modes)

$$\min \mathrm{eig}(\mathbf{X}_{\alpha\nu}, \mathbf{R}_{\mathrm{r}})$$

Efficient and accurate evaluation of $\mathbf{R}_\mathrm{r}$

The real-valued part of the impedance matrix with elements

$$Z_{pq} = \mathrm{j}kZ_0 \int_{\Omega}\int_{\Omega} \boldsymbol{\psi}_p(\boldsymbol{r}_1)\cdot\mathbf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cdot\boldsymbol{\psi}_q(\boldsymbol{r}_2)\,\mathrm{dS}_1\,\mathrm{dS}_2$$

is decomposed by expanding the Green dyadic in regular $\boldsymbol{u}_\alpha^{(1)}$ and out-going $\boldsymbol{u}_\alpha^{(4)}$ spherical vector waves

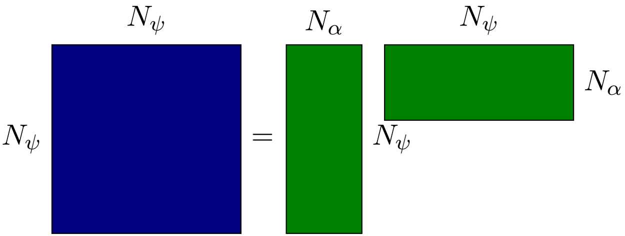
$$\mathbf{G}(\boldsymbol{r}_1,\boldsymbol{r}_2) = -\mathrm{j}k\sum_\alpha \boldsymbol{u}_\alpha^{(1)}(k\boldsymbol{r}_<)\boldsymbol{u}_\alpha^{(4)}(k\boldsymbol{r}_>),$$

where $\alpha(\tau,\sigma,m,l)$, $\boldsymbol{r}_< = \boldsymbol{r}_1$ and $\boldsymbol{r}_> = \boldsymbol{r}_2$ if $|\boldsymbol{r}_1| < |\boldsymbol{r}_2|$ and so on. Factorization $\mathbf{R}_\mathrm{r} = \mathbf{S}^\mathsf{T}\mathbf{S} = \mathbf{S}^\mathsf{H}\mathbf{S}$, where $\mathbf{S}$ has the elements

$$S_{\alpha p} = kZ_0^{1/2}\int_{\Omega}\boldsymbol{\psi}_p(\boldsymbol{r})\cdot\boldsymbol{u}_\alpha^{(1)}(k\boldsymbol{r})\,\mathrm{dS}.$$

See [Tay+17] and compare with the T-matrix [Kri16], FMM [CRW93], and far-field expansion [GN13] methods.

Properties of $\mathbf{R} = \mathbf{S}^{\mathsf{H}}\mathbf{S}$



- N_ψ and $N_\alpha \ll N_\psi$ number of basis function and spherical modes, respectively.
- $N_\alpha = 2L(L+2)$ with $L \approx \lceil ka + 3 + 7(ka)^{1/3} \rceil$, exponential convergence.
- Single (surface) integral (negligible computational cost).
- $\mathbf{R}_\mathrm{r} = \mathbf{S}^{\mathsf{H}}\mathbf{S} \succeq \mathbf{0}$ (in theory and practice).
- Radiated field expanded in spherical modes $\mathbf{F} = \mathbf{SI}$.
- Radiated power $P_\mathrm{r} = \frac{1}{2}\mathbf{I}^{\mathsf{H}}\mathbf{RI} = \frac{1}{2}|\mathbf{SI}|^2 = \frac{1}{2}\sum_\alpha |F_\alpha|^2$.

Computational efficiency

Factorize radiation matrix $\mathbf{R}_\mathrm{r} = \mathbf{S}^\mathsf{H}\mathbf{S}$ to rewrite the eigenvalue problem $\mathrm{eig}(\mathbf{X}_{\alpha\nu}, \mathbf{R}_\mathrm{r})$

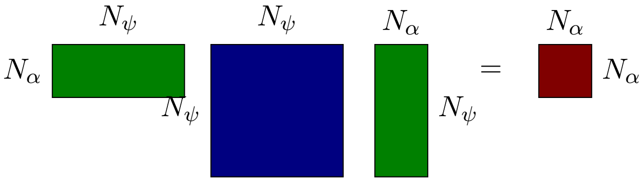
$$\mathbf{X}_{\alpha\nu}\mathbf{I} = \lambda\mathbf{R}_\mathrm{r}\mathbf{I} = \lambda\mathbf{S}^\mathsf{H}\mathbf{S}\mathbf{I}$$

as

$$\mathbf{I} = \lambda\mathbf{X}_{\alpha\nu}^{-1}\mathbf{S}^\mathsf{H}\mathbf{S}\mathbf{I} \Rightarrow \mathbf{S}\mathbf{I} = \lambda\mathbf{S}\mathbf{X}_{\alpha\nu}^{-1}\mathbf{S}^\mathsf{H}\mathbf{S}\mathbf{I}$$

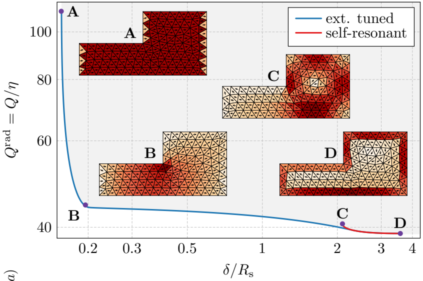
Implying an $N_\alpha \times N_\alpha$ eigenvalue problem

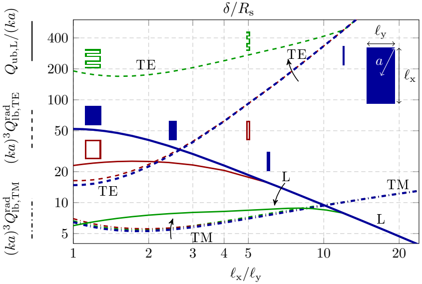
$$\lambda^{-1} = \mathrm{eig}(\mathbf{S}\mathbf{X}_{\alpha\nu}^{-1}\mathbf{S}^\mathsf{H})$$



Conclusions

- Trade-off between Q^{rad} and η
  - Big difference between optimal Q^{rad} and η for externally tuned cases.
  - Smaller for self-resonant cases with a single material (homogeneous resistivity).
  - Complex trade-off for inhomogeneous resistivity.
- Relaxation to eigenvalue problems.
- Factorization $\mathbf{R}_{\mathrm{r}} = \mathbf{S}^{\mathrm{H}}\mathbf{S}$ for computational efficiency.





References I

[CHE12] M. Capek, P. Hazdra, and J. Eichler. “A method for the evaluation of radiation Q based on modal approach”. *IEEE Trans. Antennas Propag.* 60.10 (2012), pp. 4556–4567.

[CRW93] R. Coifman, V. Rokhlin, and S. Wandzura. “The Fast Multipole Method for the Wave Equation: A Pedestrian Prescription”. *IEEE Antennas Propag. Mag.* 35.3 (1993), pp. 7–12.

[CW15] Y. Chen and C.-F. Wang. *Characteristic Modes: Theory and Applications in Antenna Engineering*. John Wiley & Sons, 2015.

[GCJ12] M. Gustafsson, M. Cismasu, and B. L. G. Jonsson. “Physical bounds and optimal currents on antennas”. *IEEE Trans. Antennas Propag.* 60.6 (2012), pp. 2672–2681.

[GN06] M. Gustafsson and S. Nordebo. “Characterization of MIMO antennas using spherical vector waves”. *IEEE Trans. Antennas Propag.* 54.9 (2006), pp. 2679–2682.

[GN13] M. Gustafsson and S. Nordebo. “Optimal Antenna Currents for Q, Superdirectivity, and Radiation Patterns Using Convex Optimization”. *IEEE Trans. Antennas Propag.* 61.3 (2013), pp. 1109–1118.

[Gus+16] M. Gustafsson, D. Tayli, C. Ehrenborg, M. Cismasu, and S. Nordebo. “Antenna current optimization using MATLAB and CVX”. *FERMAT* 15.5 (2016), pp. 1–29.

[Har60] R. F. Harrington. “Effect of Antenna Size on Gain, Bandwidth and Efficiency”. *Journal of Research of the National Bureau of Standards – D. Radio Propagation* 64D (1960), pp. 1–12.

[HM71] R. F. Harrington and J. R. Mautz. “Theory of characteristic modes for conducting bodies”. *IEEE Trans. Antennas Propag.* 19.5 (1971), pp. 622–628.

[HM72] R. F. Harrington and J. R. Mautz. “Control of radar scattering by reactive loading”. *IEEE Trans. Antennas Propag.* 20.4 (1972), pp. 446–454.

[JC17] L Jelinek and M Capek. “Optimal Currents on Arbitrarily Shaped Surfaces”. *IEEE Trans. Antennas Propag.* 65.1 (2017), pp. 329–341.

[Kri16] G. Kristensson. *Scattering of Electromagnetic Waves by Obstacles*. SciTech Publishing, an imprint of the IET, 2016.

[Pfe17] C. Pfeiffer. “Fundamental Efficiency Limits for Small Metallic Antennas”. *IEEE Trans. Antennas Propag.* 65.4 (2017), pp. 1642–1650.

[Tay+17] D. Tayli, M. Capek, L. Akrou, V. Losenicky, L. Jelinek, and M. Gustafsson. *Accurate and Efficient Evaluation of Characteristic Modes*. Tech. rep. LUTEDX/(TEAT-7258)/1–22/(2017). Lund University, 2017.

[Tha18] H. L. Thal. “Radiation efficiency limits for elementary antenna shapes”. *IEEE Trans. Antennas Propag.* 99 (2018).

[VCF10] J. Volakis, C. C. Chen, and K. Fujimoto. *Small Antennas: Miniaturization Techniques & Applications*. McGraw-Hill, 2010.