Fundamental Limits on Capacity and Degrees of Freedom for Arbitrary Shapes

Mats Gustafsson

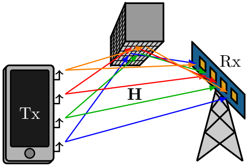
M. Gustafsson and J. Lundgren. “Degrees of Freedom and Characteristic Modes”. IEEE Antennas Propag. Mag. (2024). (in press), pp. 2–12
M. Gustafsson. “Degrees of Freedom for Radiating Systems”. arXiv:2404.08976 (2024)

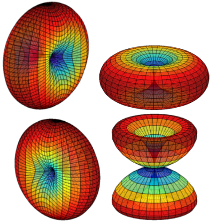
Electrical and Information Technology, Lund University, Sweden

Degrees of Freedom (DoF)

DoF is a metric of the number of parameters that can be adjusted to achieve a desired level of performance

- Spatial DoF such as modes at a fixed frequency





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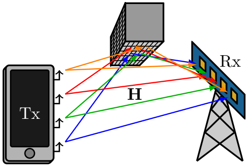

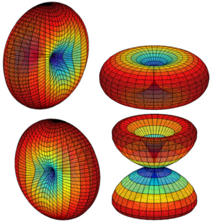

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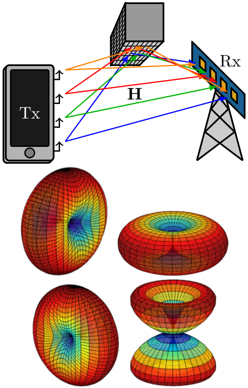

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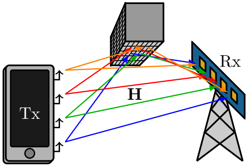

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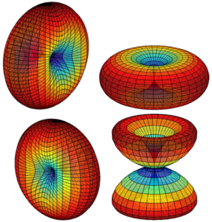

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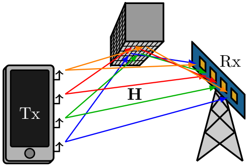

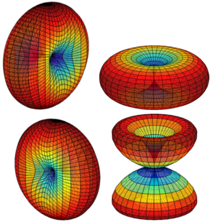

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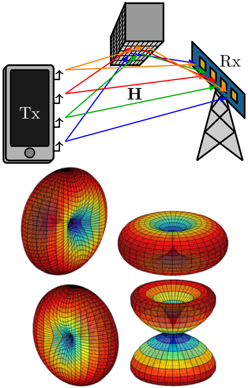

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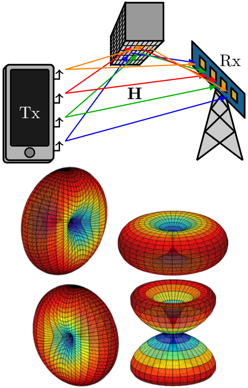
- Different NDoFs for different applications



Franceschetti, Wave theory of information (2017). Bucci and Franceschetti, On the degrees of freedom of scattered fields (1989). M. Migliore, On the role of the number of degrees of freedom of the field in MIMO channels (2006). Kildal, Martini, and Maci, Degrees of freedom and maximum directivity of antennas (2017), ...

Classical approach to NDoF of EM field (half a century ago)

Spherical radiator with radius a and surface area $A = 4\pi a^2$

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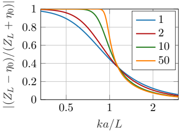

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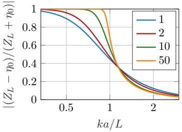

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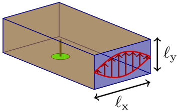
Rectangular waveguides with number of propagating modes from the cutoff wavenumbers (TE and TM cases)



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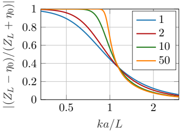

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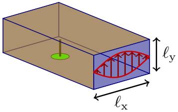

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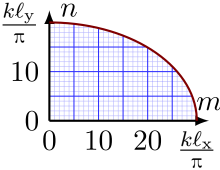

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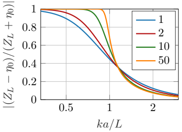

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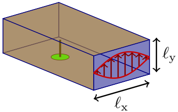

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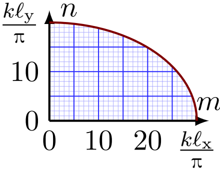

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Approximately $N \approx 2\pi A/\lambda^2$ for surface area A. Is this true for other shapes?

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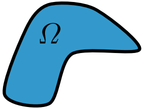
- Weyl's law describes the distributions of eigenvalues ν_n for the Laplace operator $-\nabla^2 u_n = \nu_n u_n$ in a region $\Omega \subset \mathbb{R}^d$ with Dirichlet or Neumann boundary conditions [Wey11], see [Are+09]

Hermann Weyl 1885–1955



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for a region Ω with volume $|\Omega|$ (in $\mathbb{R}^3$, area in $\mathbb{R}^2$, length in $\mathbb{R}^1$) and with $w_d = \pi^{d/2}/(d/2)!$ denoting the volume of the unit ball in $\mathbb{R}^d$

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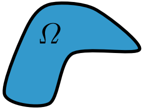

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- Number of positive eigenvalues for Helmholtz equation in Ω

$$\nabla^2 u_n + k^2 u_n = \mu_n u_n \Leftrightarrow -\nabla^2 u_n = (k^2 - \mu_n) u_n$$

  for a wavenumber $k = 2\pi/\lambda$ with wavelength λ

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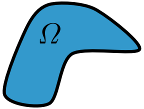

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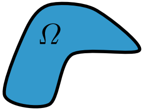

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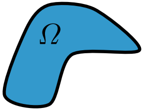

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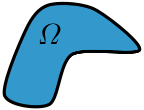

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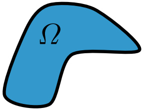

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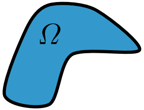

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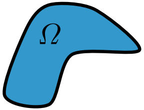

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- Times two for two polarizations in $\mathbb{R}^2$ (TE, TM waveguide modes)

Weyl's law and radiating (open) regions

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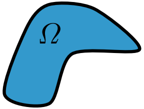

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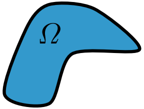

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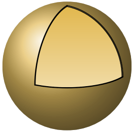

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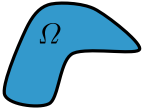

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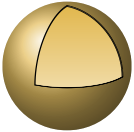

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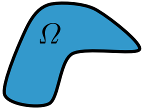

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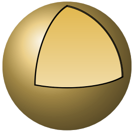

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- Surface area of the sphere and $L \approx ka$ [BI97]

$$N_{\mathrm{A}} \approx 2(ka)^2 = \frac{k^2 A}{2\pi} = \frac{2\pi A}{\lambda^2}$$

propagating modes for $ka \gg 1$ similar to Weyl's law

Weyl's law and radiating (open) regions

- Weyl's law ($\mathbb{R}^2$) for the number of propagating modes in PEC waveguides
$$N \approx \frac{k^2 A}{2\pi} = \frac{2\pi A}{\lambda^2}$$
- What about radiating structures in $\mathbb{R}^3$ with $\mathbb{R}^2$ surface?
- In contrast to waveguides, there is no clear cut-off between propagating and evanescent waves, *cf.*, spherical region with radius a.
- Surface area of the sphere and $L \approx ka$ [BI97]
$$N_{\mathrm{A}} \approx 2(ka)^2 = \frac{k^2 A}{2\pi} = \frac{2\pi A}{\lambda^2}$$
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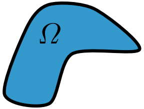
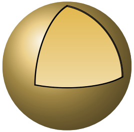
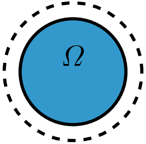
- No clear transition between propagating and evanescent modes from the expanding surface area with increasing radial distance a



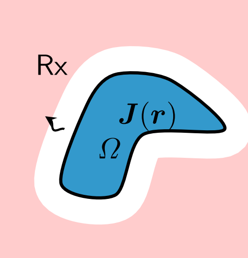




Weyl's law and radiating (open) regions

- Weyl's law can be interpreted locally as NDoF for radiation to the volume outside a slightly enlarged version of Ω

$$N \approx 2\pi A/\lambda^2$$



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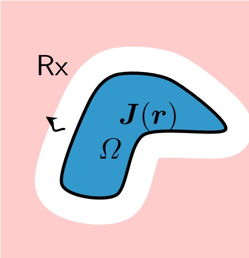

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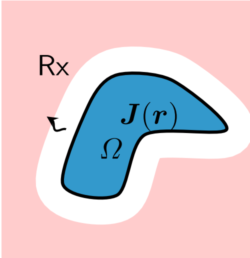

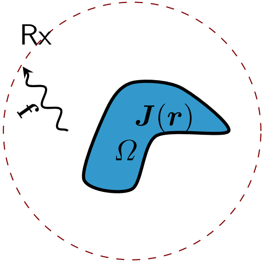

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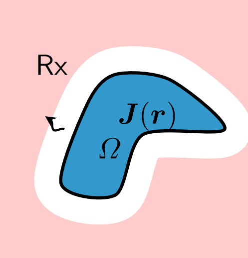

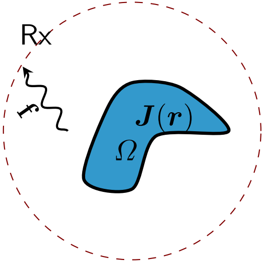

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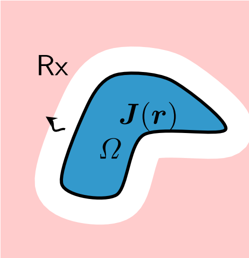
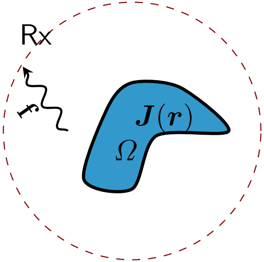
- For antenna systems often common to consider communication from a region Ω to the far field. NDoF considered here
- What does these NDoF for radiating systems depend on?
- NDoF defined as the number of significant characteristic modes (CM) $|t_n|^2 \geq 1$ (or $|\lambda_n| \leq 1$) is given by approximately [GL24]

$$\frac{N_{\mathrm{A}}}{2} = \frac{4\pi\langle A_{\mathrm{s}}\rangle}{\lambda^2} = \frac{|A_{\mathrm{s}}|}{\lambda^2} \overset{\substack{\text{convex}\\ \text{object}}}{=} \frac{\pi A}{\lambda^2}$$

where $\langle A_{\mathrm{s}}\rangle$ denotes the shadow area of the object





Average shadow (geometrical cross section) area

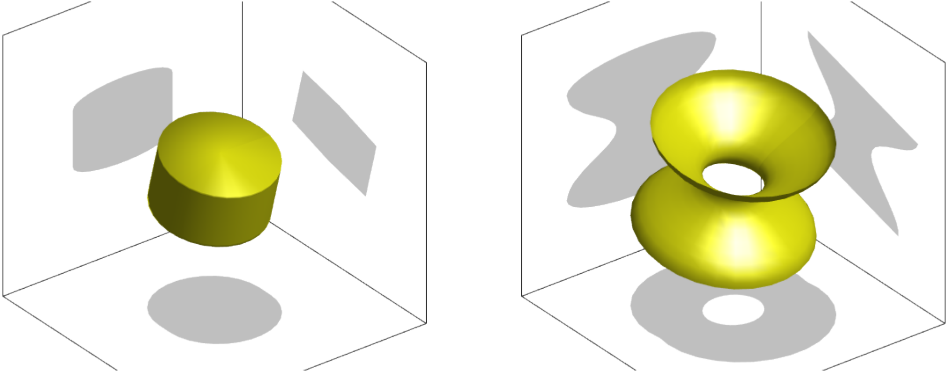


Illustration of shadow (geometrical cross section) area for convex (left) and non-convex (right) objects from three perpendicular directions. Average $\langle A_{\mathrm{s}} \rangle = \frac{1}{4\pi} \int_{4\pi} A_{\mathrm{s}}(\hat{\boldsymbol{k}})\, \mathrm{d}\Omega$

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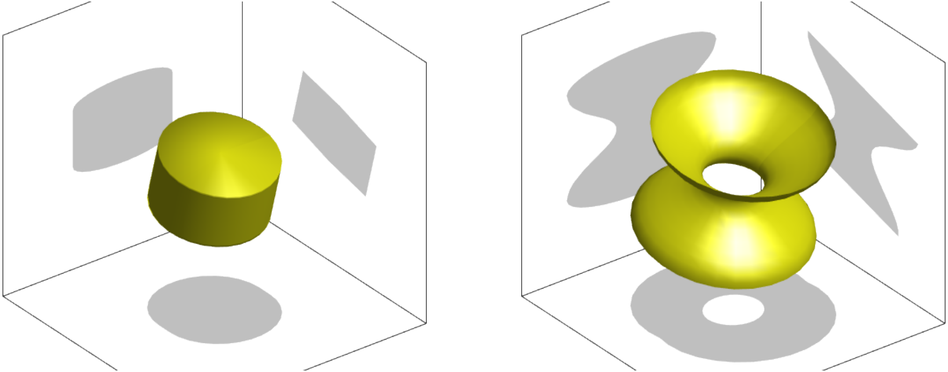

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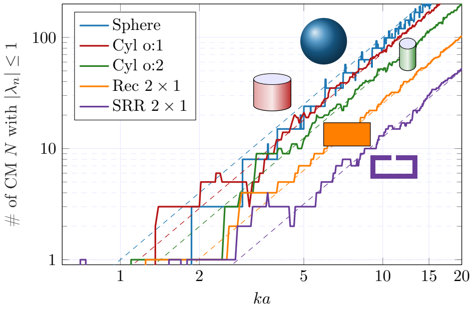
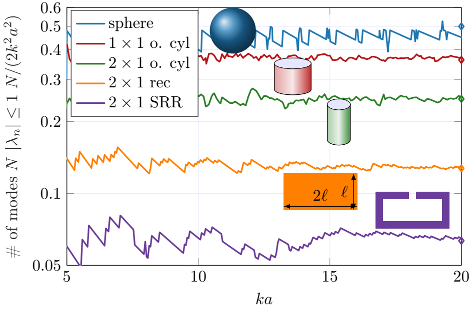
- $\langle A_\mathrm{s}\rangle = A/4$ for convex objects with surface area A (Cauchy two centuries ago)
- $\langle A_\mathrm{s}\rangle = A/2$ for planar objects (*e.g.*, PEC surface in the xy-plane)

Examples: number of CM eigenvalues with $|\lambda_n| \leq 1$





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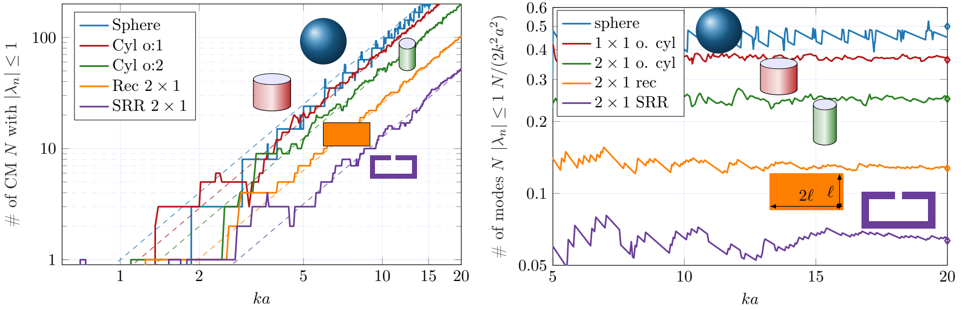

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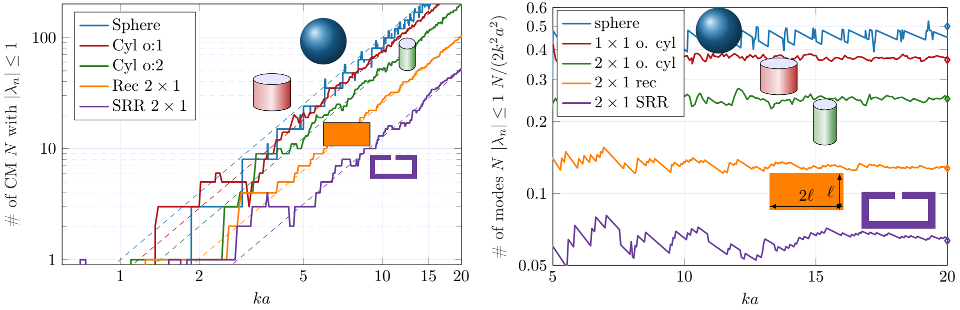

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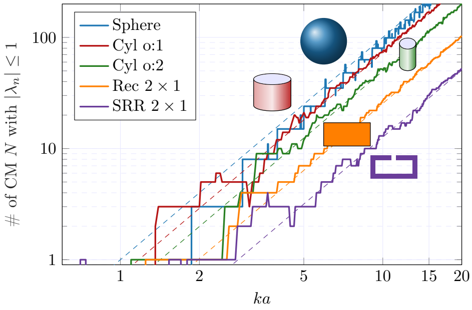

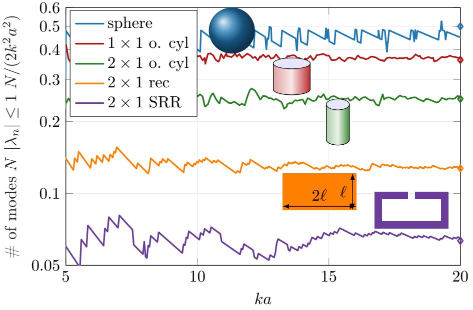

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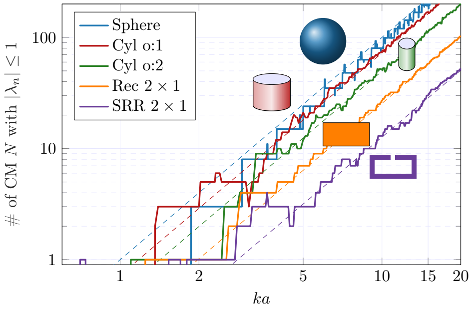

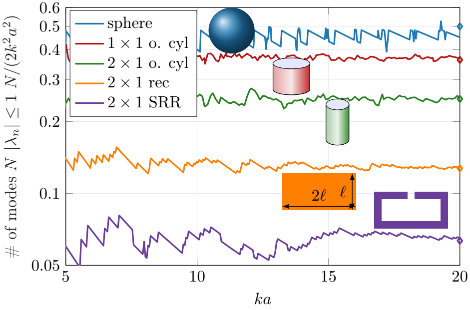

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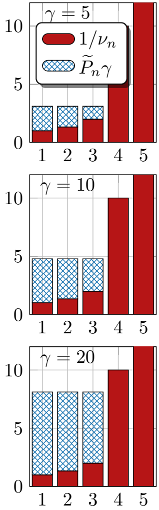
Number of significant CM ($|t_n|^2 \geq 0.5$ or $|\lambda_n| \leq 1$) can be estimated from the average shadow area $k^2\langle A_\mathrm{s}\rangle/\pi = |A_\mathrm{s}|/\lambda^2$ for $ka \gg 1$ [GL24].

Capacity limits and radiation modes

- The capacity (spectral efficiency) of the idealized system $\mathbf{f} = -\mathbf{U}\mathbf{I} + \mathbf{n}$ from [EG20]

$$\begin{aligned} &\text{maximize} && \log_2\left(\det\left(\mathbf{1} + \gamma \mathbf{U}\mathbf{P}\mathbf{U}^{\mathrm{H}}\right)\right) \\ &\text{subject to} && \mathrm{Tr}(\mathbf{R}\mathbf{P}) = 1 \\ & && \mathbf{P} \succeq \mathbf{0}, \end{aligned}$$

with the covariance matrix of the currents $\mathbf{P} = \mathcal{E}\{\mathbf{I}\mathbf{I}^{\mathrm{H}}\}$, $\mathbf{R}$ is the resistance matrix [Har68], and γ the SNR



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- Diagonalized by radiating modes. Efficiencies ν_n for waterfilling

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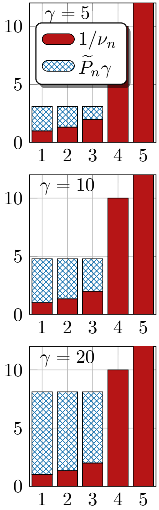

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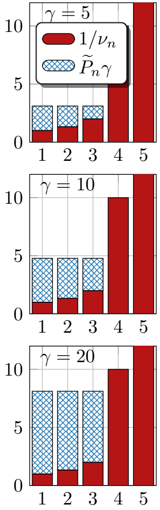
- 50% efficiency defines good modes $\nu_n \geq 0.5$



Radiation modes

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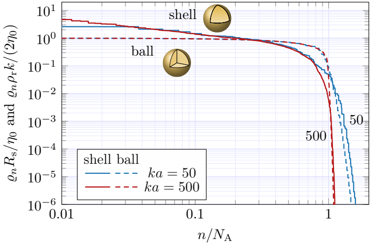

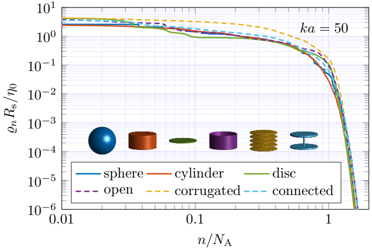

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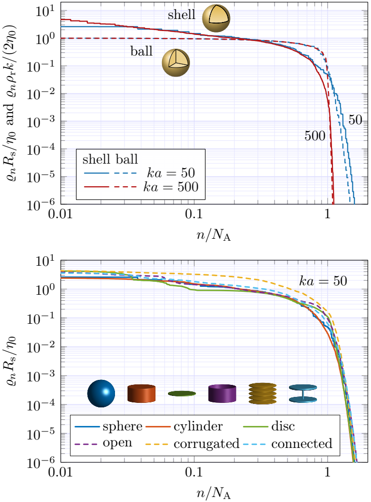

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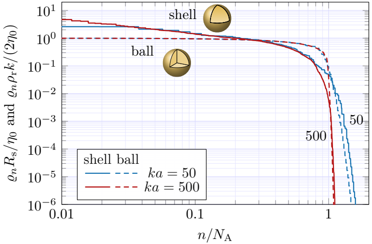

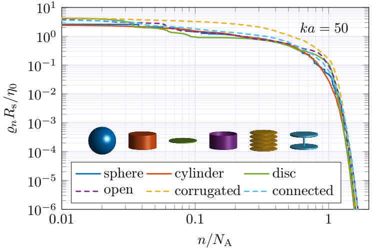

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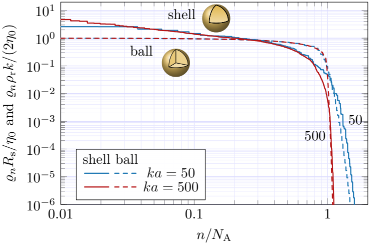

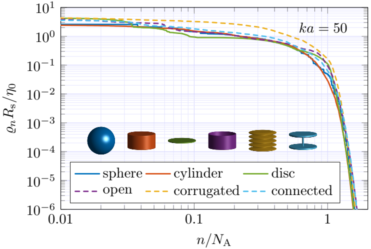

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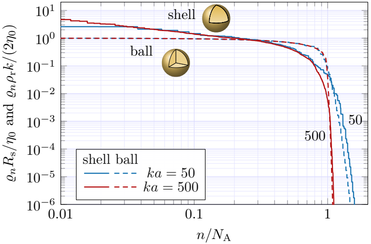
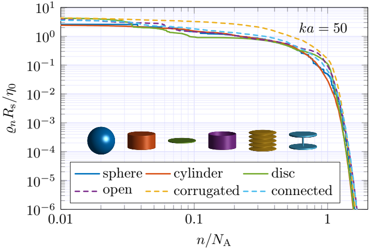
- ϱ_n decay rapidly after the N_{A}
- threshold of 50% efficiency corresponding to radiation modes $\varrho_n \geq 1$, which is here used to define a NDoF for arbitrary shaped regions and material losses [EG20]





Derivation

- The maximal partial effective area $A_{\text{eff}} = \lambda^2 G/(4\pi)$ [GC19]

$$\begin{aligned} &\text{maximize} && A_{\text{eff}} = \lambda^2 \mathbf{I}^{\mathrm{H}} \mathbf{F}^{\mathrm{H}} \mathbf{F} \mathbf{I} \\ &\text{subject to} && \mathbf{I}^{\mathrm{H}} \mathbf{R} \mathbf{I} = 1, \end{aligned}$$

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- The solution is

$$\max A_{\text{eff}} = \lambda^2 \mathbf{F}^{\mathrm{H}} \mathbf{R}^{-1} \mathbf{F} = \frac{\lambda^2}{16\pi^2} \mathbf{a}^{\mathrm{H}} \mathbf{U} \mathbf{R}^{-1} \mathbf{U}^{\mathsf{T}} \mathbf{a},$$

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- Diagonalizing using radiation modes $\tilde{\mathbf{a}} = \mathbf{Q}\mathbf{a}$, with $\sum_m Q_{nm}^2 = 1$

$$\max A_{\text{eff}} = \frac{\lambda^2}{16\pi^2}\sum_n \frac{\varrho_n|\tilde{a}_n|^2}{1+\varrho_n} = \frac{\lambda^2}{16\pi^2}\sum_{n,m} \frac{\varrho_n Q_{nm}^2|a_m|^2}{1+\varrho_n}$$

- Average of $\max A_{\text{eff}}$ over all directions and polarizations using the plane wave expansion $a_n = 4\pi \mathrm{j}^{\tau-1-l}\hat{\boldsymbol{e}}\cdot\boldsymbol{Y}_n(\hat{\boldsymbol{k}})$ [Kri16] with spherical harmonics $\boldsymbol{Y}_n$ and $\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{k}} = 0$ results in

$$\langle|a_n|^2\rangle = \frac{1}{8\pi^2}\int_{2\pi}\int_{4\pi}|a_n|^2\,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}}\,\mathrm{d}\Omega_{\hat{\boldsymbol{e}}} = 2\pi\int_{4\pi}\boldsymbol{Y}_\nu(\hat{\boldsymbol{k}})\cdot\boldsymbol{Y}_\nu(\hat{\boldsymbol{k}})\,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}} = 2\pi$$

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- Identity for the average maximal effective area expressed in ϱ_n

$$\langle\max A_{\mathrm{eff}}\rangle = \frac{\lambda^2}{8\pi}\sum_{n=1}^{\infty}\nu_n,\quad \langle\max G\rangle = \frac{1}{2}\sum\nu_n$$

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$$\langle |a_n|^2 \rangle = \frac{1}{8\pi^2} \int_{2\pi} \int_{4\pi} |a_n|^2 \,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}} \,\mathrm{d}\Omega_{\hat{\boldsymbol{e}}} = 2\pi \int_{4\pi} \boldsymbol{Y}_\nu(\hat{\boldsymbol{k}}) \cdot \boldsymbol{Y}_\nu(\hat{\boldsymbol{k}}) \,\mathrm{d}\Omega_{\hat{\boldsymbol{k}}} = 2\pi$$

- Identity for the average maximal effective area expressed in ϱ_n

$$\langle \max A_{\text{eff}} \rangle = \frac{\lambda^2}{8\pi} \sum_{n=1}^{\infty} \nu_n, \quad \langle \max G \rangle = \frac{1}{2} \sum \nu_n$$

- Typical surface resistivities of metals, such as $R_\text{s} \approx 0.01\,\Omega/\square$ for Cu, the ϱ_n approximately divide into two groups: those with $\varrho_n \gg 1$ and those with $\varrho_n \ll 1$, resulting in efficiencies ν_n according to

$$\nu_n = \frac{\varrho_n}{1+\varrho_m} \approx \begin{cases} 1 & n < N_\text{A} \\ 0 & n > N_\text{A}, \end{cases}$$

where N_A denotes the NDoF for the given shape and frequency

- Inserting using the cutoff produces the estimates

$$\langle \max A_{\mathrm{eff}} \rangle \approx \frac{\lambda^2}{8\pi} N_{\mathrm{A}} \quad \text{and } N_{\mathrm{A}} \approx 2\langle \max G \rangle$$

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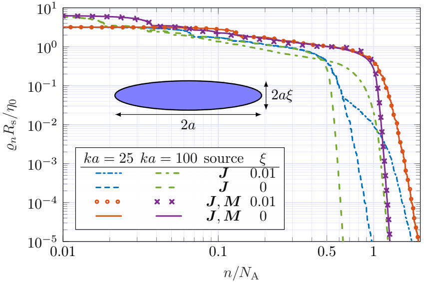
  where $|A_{\mathrm{s}}| = 4\pi \langle A_{\mathrm{s}} \rangle$ denotes the total shadow area
- NDoF is twice the number of significant characteristic modes [GL24] and identical to Weyl's law for convex shapes

Electric and magnetic currents

- Magnetic and electric currents increases the NDoF compared with the solely electric case. This is particularly evident for electrical sizes where the geometrical structure is not resolved by the wavelength



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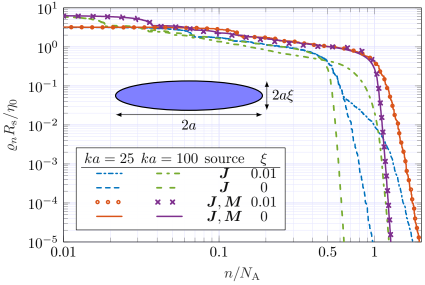

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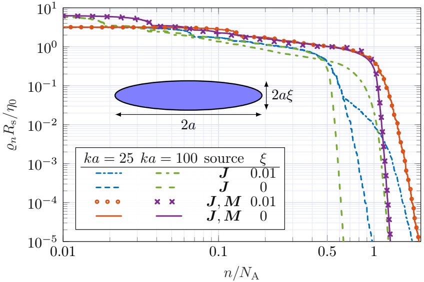

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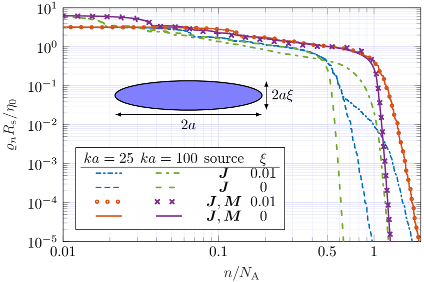
  - Considering *e.g.*, a planar region with only electric currents enforces a symmetry of the radiated field in the up and down directions reducing the NDoF [EG20]

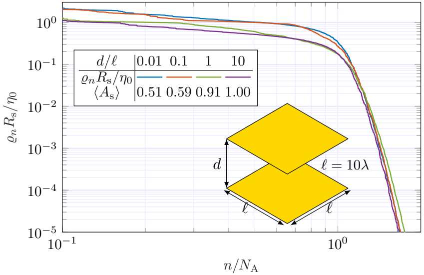


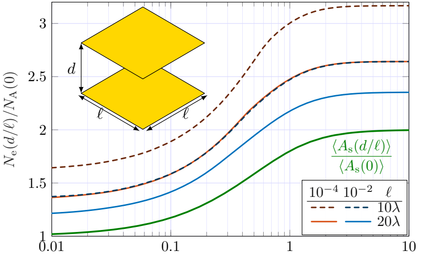
Electric and magnetic currents

- Effective NDoF [Shi+00; Yua+22] expressed in radiation mode efficiency as

$$N_{\mathrm{e}} = \frac{(\operatorname{Tr}\mathbf{H}\mathbf{H}^{\mathrm{H}})^2}{\|\mathbf{H}\mathbf{H}^{\mathrm{H}}\|_{\mathrm{F}}^2} = \frac{\left(\sum_{n=1}^{\infty}\nu_n\right)^2}{\sum_{n=1}^{\infty}\nu_n^2}.$$

  Depend on the material losses but not on a threshold level for the efficiency





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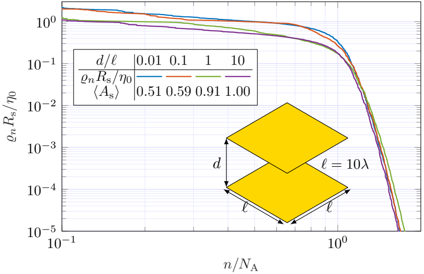

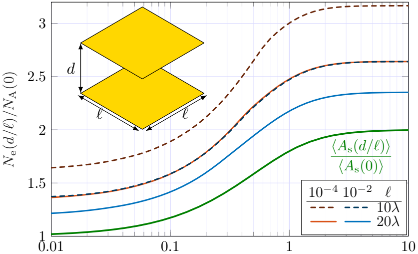

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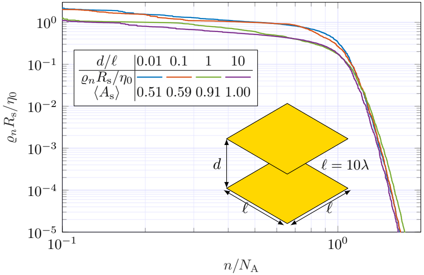

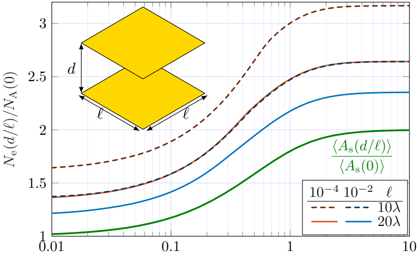

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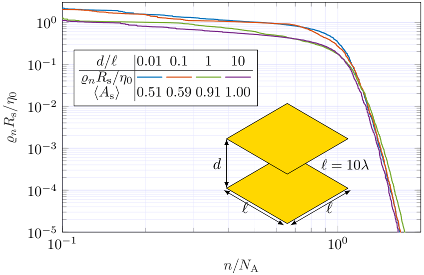

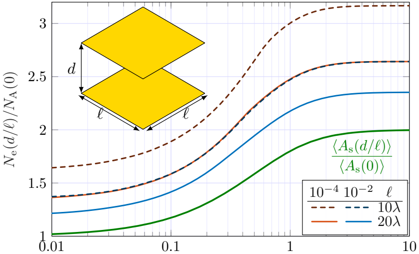

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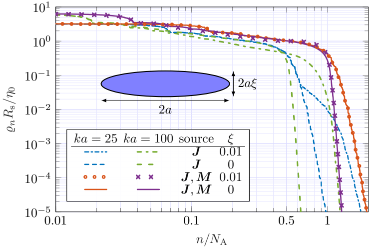

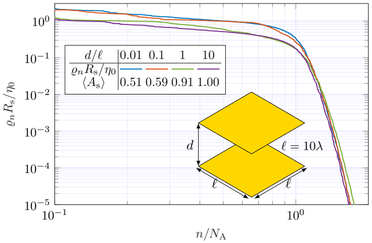

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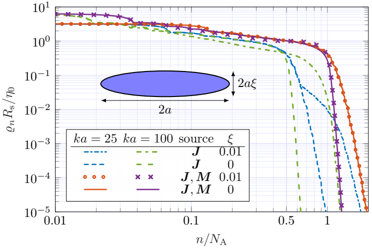

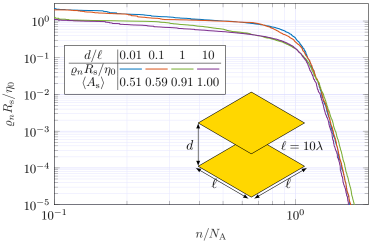

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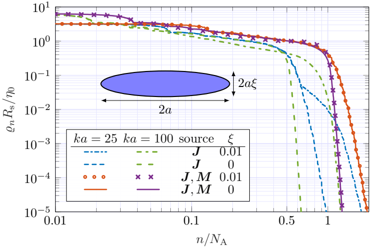

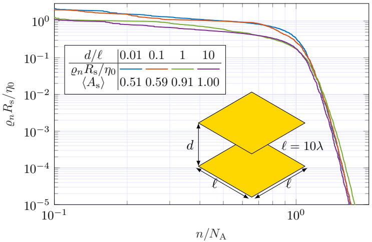

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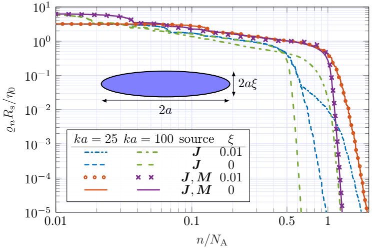

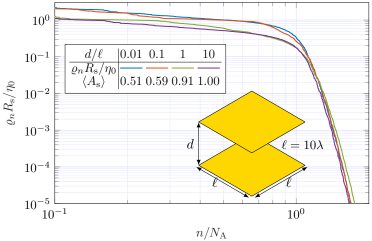

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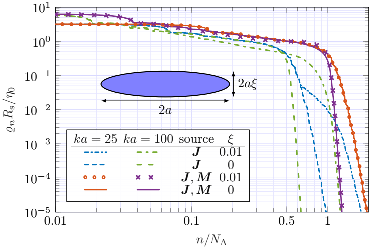

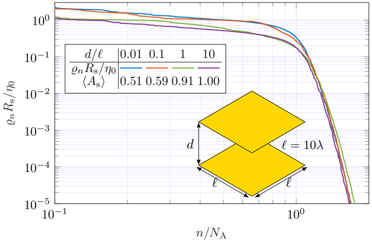

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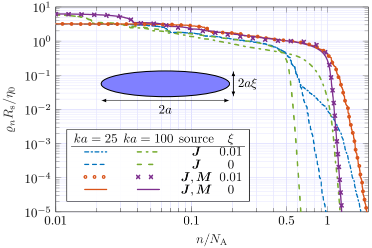
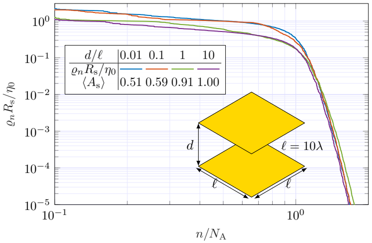
- Need electric and magnetic currents for sheets and thin structures





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